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ANALYSIS AND CONTROL OF AN UNDERACTUATED PENDULUM

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ABSTRACT

We consider a planar pendulum with a massless rope, whose length can be influenced. This forms a nonlinear underactuated system with a simple structure but challenging properties. The aim is to influence the pendulum oscillation by applying a force to the rope, i. e., by changing the length temporarily. If the time derivative of total energy is expressed as Lie derivative along the vector fields of the Byrnes-Isidori normal form, a simple rule can be deduced: For oscillation damping, the load must be lowered near the swing-through-point. To reach the the original length, hoisting has to take place near the turning point. This rule can be implemented in a controller with variable structure.

Index Terms— Pendulum, variable length, energy, second order sliding mode, double integrator

1. INTRODUCTION

Mechanical systems which have fewer actuators than degrees of freedom, so called underactuated systems, represent an interesting and challenging class from the viewpoint of control theory. The study of such systems is motivated by various reasons. In some cases the lack of actuators arises due to weight saving efforts or actuator failure. Underactuated manipulators are examples for this class.

For other systems underactuation originates because employment of all available control inputs may be unwanted. This is the case for container cranes. In principle, the crane trolley position is controlled and therefore the system is fully actuated. However, for damping payload pendulations a motion of the trolley is not desired in order to improve the crane operators comfort [1]. A similar restriction exists in the case of flying cranes [2]. In the past, various approaches have been proposed, often considering a specific crane application, e. g. see [3] and the references there.

We, however, concentrate on an abstract and thus simple model, applying techniques from modern nonlinear control theory. We investigate the control problem of influencing the oscillation of a mathematical pendulum only by controlling the rope length within bounds. Besides possible crane applications we want to contribute to the general comprehension of this system, as the pendulum features an important role in the understanding of dynamical systems in general.

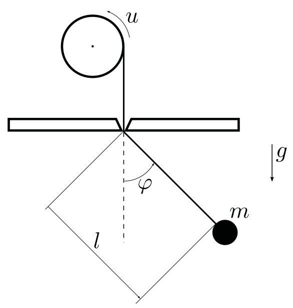

Fig. 1. Pendulum with a rope of variable length.

2. SYSTEM DYNAMICS

In this section, a mathematical model of the investigated system will be derived. Figure 1 shows the system with a hoisting drum as actuator. While this helps to illustrate the control system, we do not consider the dynamics of the motor or the inertia of the drum to keep the calculations simpler. As a further assumption, the rope is modeled as massless and inelastic. Thus, we have a classical mathematical pendulum with the additional property that its length can be controlled by a force which is applied to the rope. Its kinetic energy $E_{\text{kin}} = \frac{m}{2}(\dot{l}^2 + \dot{\varphi}^2 l^2)$ and potential energy $E_{\text{pot}} = mg(l_0 - l\cos\varphi)$ together compose the Lagrangian $L := E_{\text{kin}} - E_{\text{pot}}$. Now, the equations of motion can be deduced by means of the Lagrangian formalism. As its result one obtains

$$\begin{aligned} \ddot{l} &= l\dot{\varphi}^2 + g\cos\varphi - \tfrac{u}{m}, \\ \ddot{\varphi} &= \tfrac{-2\dot{l}\dot{\varphi}}{l} - \tfrac{g}{l}\sin\varphi, \end{aligned}$$

where u is the force applied to the rope.

As next step the state vector $\mathbf{x} = (l, \dot{l}, \varphi, \dot{\varphi})^T$ is introduced, which leads to the state space form

$$\dot{\mathbf{x}} = \underbrace{\begin{pmatrix} x_2 \\ x_1 x_4^2 + g\cos x_3 \\ x_4 \\ -2\frac{x_2 x_4}{x_1} - \frac{g}{x_1}\sin x_3 \end{pmatrix}}_{=: \tilde{\mathbf{f}}(\mathbf{x})} + \underbrace{\begin{pmatrix} 0 \\ -\frac{1}{m} \\ 0 \\ 0 \end{pmatrix}}_{=: \tilde{\mathbf{g}}(\mathbf{x})} u$$

of the system dynamics.

The physical background of that system motivates

Assumption A: $\quad x_3 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $x_1 > 0$.

This implies that all equilibrium points are given by the set $\{\mathbf{x} \in \mathbb{R}^4 | x_1 > 0, x_2 = x_3 = x_4 = 0\}$. In other words, we have no isolated equilibrium points but an one-dimensional manifold. It is easy to see that the linearization of the system about an arbitrary equilibrium point is not controllable because the input does not affect the x_3-x_4-subsystem.

Now the question arises if it is possible to perform input-state linearization [4, Sect. 4.2]. The answer can be found by some differential geometric considerations. Recall that the Lie bracket of two vector fields $\tilde{\mathbf{f}}, \tilde{\mathbf{g}}$ is defined by $[\tilde{\mathbf{f}}, \tilde{\mathbf{g}}] = \tilde{\mathbf{g}}'\tilde{\mathbf{f}} - \tilde{\mathbf{f}}'\tilde{\mathbf{g}}$, where $\tilde{\mathbf{f}}'$ and $\tilde{\mathbf{g}}'$ denote the Jacobians of $\tilde{\mathbf{f}}$ and $\tilde{\mathbf{g}}$, respectively. Higher order Lie brackets are defined recursively by $\mathrm{ad}_{\tilde{\mathbf{f}}}^{k+1}\tilde{\mathbf{g}} = [\tilde{\mathbf{f}}, \mathrm{ad}_{\tilde{\mathbf{f}}}^{k}\tilde{\mathbf{g}}]$ with $\mathrm{ad}_{\tilde{\mathbf{f}}}^{0}\tilde{\mathbf{g}} = \tilde{\mathbf{g}}$. To achieve input-state linearization the distribution $\Delta := \mathrm{span}\{\tilde{\mathbf{g}}, \mathrm{ad}_{\tilde{\mathbf{f}}}\tilde{\mathbf{g}}, \mathrm{ad}_{\tilde{\mathbf{f}}}^2\tilde{\mathbf{g}}\}$ has to be involutive [4, Th. 4.2.3]. Unfortunately, we have $[\mathrm{ad}_{\tilde{\mathbf{f}}}\tilde{\mathbf{g}}, \mathrm{ad}_{\tilde{\mathbf{f}}}^2\tilde{\mathbf{g}}] \notin \Delta$, i. e., it is not possible to obtain a linear controllable system by choosing a suitable feedback law. In other words: There exists no scalar function defined on the state-space for which the system (2) has relative degree $r = 4$. Dealing with a single-input system we can also conclude that the system is not differentially flat [5].

Based on similar considerations as above we can show that the maximum achievable relative degree is 3 [4, Th. 4.8.2]. In fact, system (2) has the maximum relative degree 3 using the component x_3 of the state vector $\mathbf{x}$ as the output.

In the following section a *partial* feedback linearization will be performed instead, which will be the fundament of the controller design in Section 4.

3. BYRNES-ISIDORI NORMAL FORM AND POWER BALANCE

If the rope length x_1 is considered as output, the system has relative degree of 2. With the feedback law

$$u = m(x_1 x_4^2 + g\cos x_3 - v),$$

a new input $v = \ddot{x}_1$ can be introduced. This is known as input-output-linearization [4, Sect. 4.1]. After this partial linearization the system equations read as follows:

$$\dot{\mathbf{x}} = \underbrace{\begin{pmatrix} x_2 \\ 0 \\ x_4 \\ -2\frac{x_2 x_4}{x_1} - \frac{g}{x_1}\sin x_3 \end{pmatrix}}_{=: \mathbf{f}(\mathbf{x})} + \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{=: \mathbf{g}(\mathbf{x})} v.$$

This is the Byrnes-Isidori normal form as the nonlinear subsystem is independent from the input v.

The total energy of the system,

$$\begin{aligned} E &:= E_{\text{kin}} + E_{\text{pot}} \\ &= \tfrac{m}{2}(x_2^2 + x_4^2 x_1^2) + mg(l_0 - x_1\cos x_3), \end{aligned}$$

depends on the state. It can be considered as a scalar field which maps from the state space to the real numbers. Its Lie derivative along a vector field $\mathbf{f}$ is denoted $L_{\mathbf{f}}E$ and is defined by the scalar product $E'(\mathbf{x})\mathbf{f}(\mathbf{x})$, where E' is the gradient of E. Thus, its time derivative can be expressed using the vector fields $\mathbf{f}$ and $\mathbf{g}$ of the system dynamics:

$$\begin{aligned} \dot{E} &= L_{\mathbf{f}}E + L_{\mathbf{g}}Ev \\ &= -x_2 \underbrace{mg\cos x_3}_{T_1} - x_2 \underbrace{m x_1 x_4^2}_{T_2} + \underbrace{m x_2 v}_{T_3}. \end{aligned} \quad (1)$$

Now, we introduce a further reasonable assumption

Assumption B:

$$x_2(t_0) = x_2(t_e) = 0, \; v(t_0) = v(t_e) = 0, \quad (2)$$

which means that only transitions between stationary oscillation regimes are considered. In the controller design in Section 4 this is respected because at the beginning ($t = t_0$) and at the end ($t = t_e$) of each maneuver x_2 and v vanish.

Using integration by parts, Eq. (2) and the relation $v = \dot{x}_2$, it is easy to see that the contribution of $T_3 = L_{\mathbf{g}}Ev$ to the total energy is zero:

$$\begin{aligned} &\int_{t_0}^{t_e} x_2(t)v(t)dt = \\ &\quad \underbrace{[x_2(t)x_2(t)]_{t_0}^{t_e}}_{=:0} - \int_{t_0}^{t_e} \underbrace{\dot{x}_2(t)}_{v(t)} x_2(t)dt = 0. \end{aligned}$$

As consequence of Assumption A we have

$$T_1 \geq 0, \qquad T_2 \geq 0$$

and therefore

$$\operatorname{sign} L_{\mathbf{f}}E(\mathbf{x}) = -\operatorname{sign}(x_2) \quad (3)$$

holds. In other words: decreasing the rope length increases the energy while increasing the rope length causes energy loss of the system.

The easiest way to influence the total energy would obviously be to change x_1 only in one direction. However, in nearly all applications there will be lower and upper bounds of admissible values for x_1. This motivates a control approach consisting of alternating hosting and lowering maneuvers. For oscillation damping, hoisting ($x_2 < 0$) has to take place when $T_1 + T_2$ is small, whereas lowering of the load should happen when $T_1 + T_2$ is big. For amplification, the role of hoisting and lowering has to be swapped. Both T_1 and T_2 change during an oscillation cycle of the pendulum. It follows from (1) that both terms reach their maximum when the pendulum passes the vertical line under its mounting point, i. e., $x_3 = 0$. On the other hand, both terms encounter their minimum when the pendulum changes its direction, i. e., $x_4 = 0$.

Hence, it is in principle clear when x_2 should take which value. In the following a control scheme will be developed to implement this approach.

4. VARIABLE STRUCTURE CONTROL

The aforementioned rule can be implemented by restricting the system dynamics to a submanifold of the state space. A similar approach has already been applied to an underactuated manipulator [6]. In this section we derive a control law which can be used for oscillation damping. As mentioned, amplifying can be achieved analogously.

4.1. Lowering controller in the x_1-x_3-plane

We consider the x_1-x_3-plane, which is a two dimensional subspace of the state space. In this projection the free motion of the pendulum, i. e., when $x_1(t) = x_{1,0}$ and $x_2(t) \equiv 0$, shows up as a vertical line at $x_{1,0}$ bounded from $-x_{3,\text{max}}$ to $x_{3,\text{max}}$. However, to achieve damping the pendulum has to pass the swing-through-point with $x_2 > 0$. This implies an increasing cable length x_1, i. e., its trajectory would cross the x_1-axis either from the lower left to the upper right or from the upper left to the lower right. This behavior could be achieved if the system dynamics is stabilized to a suitable sliding surface.

It is natural to fix minimum and maximum rope length $x_{1,\text{min}}$ and $x_{1,\text{max}}$. Furthermore we define

$$\begin{aligned} x_{1,0} &:= \tfrac{1}{2}(x_{1,\text{max}} + x_{1,\text{min}}), \\ \Delta x_1 &:= \tfrac{1}{2}(x_{1,\text{max}} - x_{1,\text{min}}). \end{aligned} \tag{4}$$

To describe the sliding surface we choose a piecewise polynomial approach. In particular, we define a function ψ which yields the x_1 value depending on x_3. This function has to fulfill

$$\psi_\sigma(x_3) = \begin{cases} x_{1,0} - \sigma\Delta x_1, & \text{if } x_3 \le -\Delta x_3, \\ x_{1,0} + \sigma\Delta x_1, & \text{if } x_3 \ge \Delta x_3, \end{cases} \tag{5}$$

where $\Delta x_3 > 0$ and $\sigma \in \{-1, 1\}$ are parameters which will be fixed in Section 4.3. Near the x_1-axis, i. e., for $-\Delta x_3 < x_3 < \Delta x_3$, ψ is chosen to be a polynomial with degree of five. In order to ensure that ψ is twice continuously differentiable, it has to meet six conditions. From them the polynomial coefficients are uniquely determined. As a further consequence, the graph of ψ is symmetrical w.r.t. the point $(x_{1,0}, 0)$ in the x_1-x_3-plane and we have $\psi(0) = x_{1,0}$. The two possible realizations are shown in Figure 2.

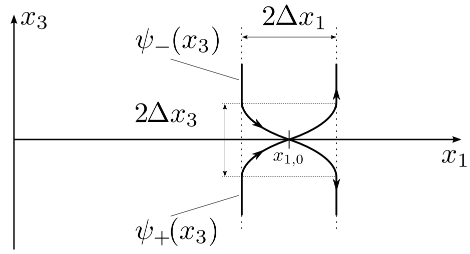

Fig. 2. Visualization of the sliding surface given by $x_1 - \psi_\sigma(x_3) = 0$ for $\sigma = +1$ and $\sigma = -1$. The desired motion on the sliding surface is indicated by arrows.

The sliding surface can now be described by the equation

$$s_\psi(\mathbf{x}) = x_1 - \psi(x_3) \overset{!}{=} 0.$$

This means if $s_\psi(\mathbf{x})$ can be stabilized to 0 the system is stabilized to the sliding surface and therefore the desired motion takes place.

A simple calculation reveals that the relative degree of s_ψ is two, i. e., we have a sliding mode problem of second order:

$$\begin{aligned} \dot{s}_\psi(\mathbf{x}) &= x_2 - \psi'(x_3)x_4, \\ \ddot{s}_\psi(\mathbf{x}) &= v - \psi''(x_3)x_4^2 \\ &\quad + \psi'(x_3)\tfrac{2x_2x_4 + g\sin x_3}{x_1}. \end{aligned}$$

The state feedback resulting from the second equation to the input v of system (3) is denoted by v_ψ and reads as follows:

$$v_\psi := w_\psi + \psi''(x_3)x_4^2 - \psi'(x_3)\left(\frac{2x_2x_4}{x_1} + \frac{g}{x_1}\sin x_3\right).$$

This feedback law performs another exact linearization step by introducing the new input w_ψ. Basing on this formulation the second order sliding mode problem can be reduced to the known problem of stabilizing the double integrator

$$\ddot{s}_\psi(\mathbf{x}) = w_\psi$$

to the origin. Several approaches are possible to achieve this goal. For example, a finite time stabilizing feedback law is proposed in [7]. For the sake of simplicity we use a PD compensator with high gain,

$$w_\psi = -k_p s_\psi(\mathbf{x}) - k_d \dot{s}_\psi(\mathbf{x}), \tag{6}$$

which can easily be realized if the state $\mathbf{x}$ is known.

To achieve energy loss the right sliding surface must be active at the right time. In Section 4.3 the law, when to switch sliding surfaces and what value to choose for σ, is given.

We now have established a control law for the lowering submaneuver of the overall maneuver. In order to keep the cable length smaller than $x_{1,\max}$, a hoisting maneuver must be performed before the next lowering can take place for further oscillation damping.

4.2. Hoisting controller in the x_1-x_4-plane

As stated, the ideal conditions for hoisting are encountered when $x_4 = 0$, i. e., when the (inevitable) amplification of oscillation is minimal. A control law which performs hoisting during the change of the pendulum's direction can be constructed similar to the above one. The pendulum's motion is now considered in the x_1-x_4-plane. The projection of an desired trajectory crosses the x_1-axis either from the lower right to the upper left or form the upper right to the lower left.

Again, we constrain the system motion to a suitable sliding surface to obtain this behavior. It can be constructed in an analogous fashion. The piecewise polynomial function is now called φ and has to fulfill

$$\varphi(x_4) = \begin{cases} x_{1,0} - \sigma\Delta x_1, & \text{if } x_4 \leq -\Delta x_4, \\ x_{1,0} + \sigma\Delta x_1, & \text{if } x_4 \geq \Delta x_4. \end{cases}$$

We require φ to be twice continuously differentiable. For $-\Delta x_4 < x_4 < \Delta x_4$ the function φ has to be a polynomial with degree of five. Then, the coefficients are also uniquely determined.

Now the sliding surface for the hoisting maneuver can be described by

$$s_\varphi(\mathbf{x}) := x_1 - \varphi(x_4) \overset{!}{=} 0.$$

It is depicted in Figure 3.

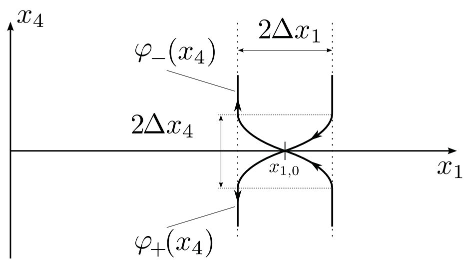

Fig. 3. Visualization of the sliding surface given by $x_1 - \psi_\sigma(x_4) = 0$ for $\sigma = +1$ and $\sigma = -1$.

As above, s_φ has relative degree of two. However, the linearizing feedback law is now slightly more complex because the input v occurs at two places in the second derivative of s_φ:

$$\begin{aligned} \dot{s}_\varphi &= x_2 - \varphi'(x_4) f_4(\mathbf{x}), \\ \ddot{s}_\varphi &= v - \varphi''(x_4)(f_4(\mathbf{x}))^2 \\ &\quad -\varphi'(x_4)(L_\mathbf{f} f_4(\mathbf{x}) + L_\mathbf{g} f_4(\mathbf{x}) v. \end{aligned}$$

In this formula f_4 denotes the fourth component of the vector field $\mathbf{f}$ from (3), i. e., the expression which determines $\dot{x}_4$.

The feedback law

$$v_\varphi := \frac{w_\varphi + \varphi''(x_4)(f_4(\mathbf{x}))^2 + \varphi'(x_4) L_\mathbf{f} f_4(\mathbf{x})}{1 - \varphi'(x_4) L_\mathbf{g} f_4(\mathbf{x})} \quad (7)$$

then introduces the new input w_φ for which

$$\ddot{s}_\varphi = w_\varphi \quad (8)$$

holds. For v_φ to be well-defined the denominator of (7) must be strictly positive which implies:

$$\begin{aligned} \varphi'(x_4) \underbrace{L_\mathbf{g} f_4(\mathbf{x})}_{-2\frac{x_4}{x_1}} &< 1 \\ \Rightarrow \frac{\varphi'(x_4)}{\varphi(x_4)} x_4 &> -\tfrac{1}{2}. \end{aligned}$$

This has to be ensured when the parameters $x_{1,0}$, Δx_1 and Δx_4 are chosen.

To stabilize the double integrator (8) we also use a high gain PD-controller

$$w_\varphi = -k_p s_\varphi(\mathbf{x}) - k_d \dot{s}_\varphi(\mathbf{x}). \quad (9)$$

4.3. Top level controller

Until now we have described a controller for every of the four submaneuvers, which are associated with φ_-, φ_+, ψ_- and ψ_+ respectively. To perform a total maneuver, after which the rope length is the same as before but the system energy has decreased, these submaneuvers have to be executed at the right time and in the right order. This can be achieved by a top level controller which activates the appropriate sliding mode controller depending on the state. Furthermore this controller has to determine the values of Δx_3 and Δx_4 in every submaneuver, due to the changing amplitude of the oscillation.

Activation of a new sliding surface has to take place when a submaneuver is finished. Thereby it is considered the end of a hoisting maneuver when the linear part of the sliding surface is reached, i. e.,

$$\sigma x_4 > \Delta x_4. \quad (10)$$

On the other hand, the lowering maneuver is considered as finished if

$$\sigma x_3 > \Delta x_3 \gamma \quad (11)$$

holds. The parameter $\gamma \in (0, 1]$ has to be introduced to adjust the time which is consumed by the lowering submaneuver. If $\gamma = 1$ the linear part of ψ is reached which needs much more than the half of $\sqrt{2\pi \frac{x_{1,0}}{g}}$. Hence, the time which is left for hoisting is short which leads to high actuator activity. Therefore $\gamma = 0.8$ is chosen. This causes the two submaneuvers to consume about the same time and thus reduces actuator activity.

Once, the end of one submaneuver has been detected the next one has to start immediately. For the new sliding surface the easiest approach for choosing Δx_3 or Δx_4, respectively, is

$$\Delta x_i := x_i(T_{\text{switch}}), \qquad i \in \{3, 4\}, \tag{12}$$

where T_{switch} denotes the instant of the submaneuver switch. Regarding the choice of the sign, the following rule can be stated:

$$\sigma := \begin{cases} -\operatorname{sign}(x_3), & \text{for } s_\psi, \\ \operatorname{sign}(x_4), & \text{for } s_\varphi. \end{cases}$$

For the initialization of the controller we assume that the pendulum is in the right turning point, i. e., $x_3 > 0$ and $x_4 = 0$. Then the first submaneuver is hoisting. As x_4 is going to decrease immediately, it is clear that $\sigma = 1$ must be chosen to obtain the right branch of the sliding surface. Because no other maneuver has been executed before, the value of Δx_4 can not be obtained by (12). It can, however, easily be calculated from energy considerations.

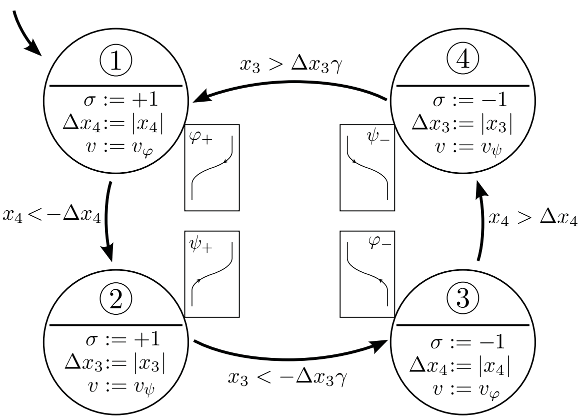

Fig. 4. Top level controller as finite state machine.

Figure 4 summarizes the logic of the top level controller as a state machine. Each of the four states represents a submaneuver whose sliding surface is depicted schematically. In the transition conditions from (10) and (11) the appropriate value of σ is used.

5. SIMULATION RESULTS

In this section we present results obtained by numerical simulation. The control scheme given in the previous section is thereby compared to a PD-controller which is designed using classical methods.

5.1. PD-controller for comparison

The PD controller has to be equipped with an additional static term to compensate the weight of the load mass,

$$u = u_{\text{PD}}(\mathbf{x}) := mg + k_0 m(x_1 - x_{1,0}) + k_1 m x_2. \tag{13}$$

This control law can be interpreted as linear spring with stiffness $k_0 m$ and viscous friction $k_1 m$.

The local asymptotic stability of the equilibrium $\mathbf{x}_0 := (x_{1,0}, 0, 0, 0)^T$ of the closed loop system given by (2) and (13) can easily be proven. From the spring analogy we can construct a Lyapunov function

$$V = E_{\text{kin}} + g x_1 (1 - \cos x_3) + \frac{k_0}{2}(x_1 - l_0)^2,$$

which is proportional to the total energy of the system, including the virtual spring. For the controlled system defined by the vector field $\bar{\mathbf{f}}(\mathbf{x}) := \tilde{\mathbf{f}}(\mathbf{x}) + \tilde{\mathbf{g}}(\mathbf{x}) u_{\text{PD}}(\mathbf{x})$ we obtain

$$\dot{V} = L_{\bar{\mathbf{f}}} V = -k_1 x_2^2,$$

whereof the asymptotic stability follows by applying La Salle's Theorem [8, Theorem 3.4].

The damping rate depends on the choice of k_0 and k_1. As stated above, the Taylor linearization about any equilibrium point is not controllable. Hence most standard design procedures for PD-controllers, e. g. as described in [9], are not applicable. One systematic way to obtain values for these parameters is to minimize the amplitude of the pendulum oscillation after a sufficient time, e. g. 60s. This yields $k_0 \approx 39$ and $k_1 \approx 0.2$

However, theses values lead to rather high amplitudes of x_1 which may not be desirable. Therefore, the optimization is restricted by the constraint:

$$\max_{t>0} |x_1(t) - x_{1,0}| \overset{!}{\leq} \Delta x_1,$$

such that both controllers are allowed to use the same "active rope length". Then $k_1 \approx 0.5$ is obtained instead.

5.2. Comparison of the two controllers

For all simulations, the following initial conditions and system parameters were used: $x_1(0) = x_{1,0} = 1\text{m}$, $x_2(0) = 0\text{ms}^{-1}$, $x_3(0) = 10°$, $x_4(0) = 0\text{s}^{-1}$, $m = 1\text{kg}$. For the sliding surfaces the shape parameter $\Delta x_1 = 0.04\text{m}$ was chosen. The subordinate PD-controllers (6) and (9) operate with $k_p = 6000$ and $k_d = 250$.

Figure 5 shows the simulation results for both the sliding-mode- (solid) and PD-controller (dashed). Two observations can be made: Both controllers synchronize themselves to the pendulum oscillation. At the beginning the PD-controller almost acts like the sliding mode controller. However, when the oscillation has already reduced its amplitude, the PD-controller causes

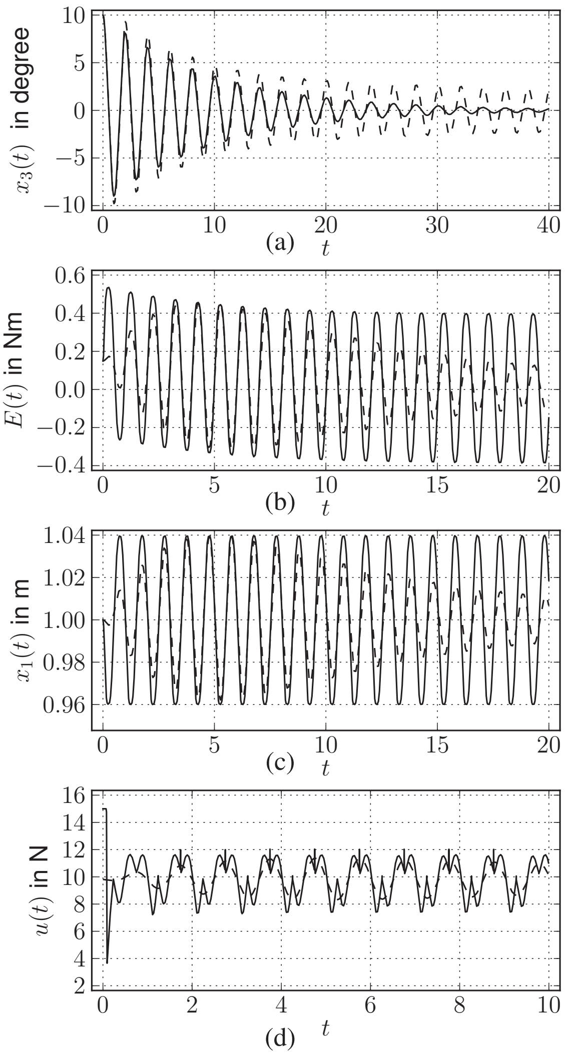

Fig. 5. Time behavior of pendulum angle (a), energy (b), rope length (c) and control force (d) for sliding-mode- (solid) and PD-controller (dashed).

reduced actor activity and therefore only little further damping. On the other hand, the sliding mode controller keeps the actor activity nearly constant which results in a much stronger damping. Figure 6 shows the trajectories temporarily restricted to the sliding surfaces from Figures 2 and 3.

6. CONCLUSION

A simple model of a pendulum with variable length was studied. It was shown that neither linearization about an equilibrium nor input-state linearization by means of nonlinear feedback can be applied successfully. Therefore an sliding mode approach basing on energy considerations has been proposed and its effectiveness has been shown by numerical simulations.

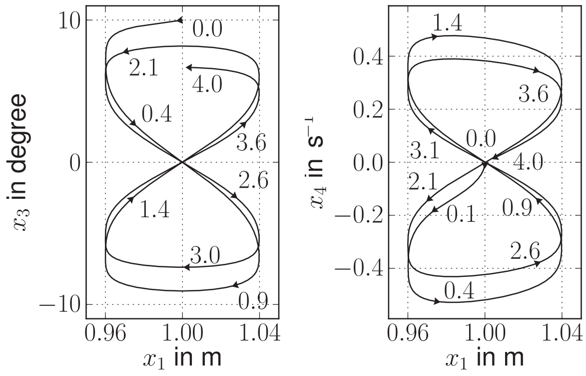

Fig. 6. Trajectory projected to x_1-x_3- and x_1-x_4-plane. Because of clarity only the first four seconds are shown.

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